LECTURE 7:

PROOF OF THE QUANTITATIVE MAIN THM. for $C^m(\mathbb{R}^n)$.

RECALL THE

QUANTITATIVE MAIN THM

("QMT")

Setting

- Fix $m, n \geq 1$.
- We work in $C^m(\mathbb{R}^n)$.
- k = large enough integer const. determined by m, n

(It may change, e.g. when we apply Helly's Thm as in a previous Lecture)

C, c, C' etc. DENOTE CONSTS.

DEPENDING ONLY ON m, n.

(These symbols may denote different constants in different occurrences.)

RECALL THAT $J_x(F)$ denotes the $m^{\underline{th}}$ DEGREE TAYLOR POLY of F at x.

(<u>NOT</u> the $(m-1)^{\text{rst}}$ DEGREE POLY)

$\mathcal{P}$ = THE VECTOR SPACE OF ALL $m^{\underline{th}}$ DEGREE POLYS.

Statement of QMT

Let $\mathcal{H} = (H(x))_{x \in E}$ be a Glaeser-stable bundle with non-empty fibers. Let $M > 0$.

If $\|\mathcal{H}\| \leq M$, then $\mathcal{H}$ has a section F with C^m-norm $\leq CM$.

Recall that $\|\mathcal{H}\| \leq M$ means:

Given $x_1, \dots, x_k \in E$,
there exist $P_1 \in H(x_1), \dots, P_k \in H(x_k)$
s.t.

$$|\partial^\alpha P_i(x_i)| \leq M$$

and

$$|\partial^\alpha (P_i - P_j)(x_i)| \leq M\,|x_i - x_j|^{m-|\alpha|}$$

for $|\alpha| \leq m$, $i, j = 1, \dots, k$.

RECALL THAT A <u>SECTION</u>

of $\mathcal{H} = (H(x))_{x \in E}$

is a fn $F \in C^m(\mathbb{R}^n)$

s.t.

$J_x(F) \in H(x)$ for all $x \in E$.

Today, we sketch the proof of QMT.

Plan of the Proof (Version 1.0)

We will partition E into finitely many **STRATA**,

$E_1, E_2, \cdots, E_\Lambda$.

$$\boxed{\text{On each } E_\nu, \; \dim H(x) \text{ is constant.}}$$

On the **Lowest Stratum** E_1,

$\dim H(x)$ is as low as possible,

i.e.,

$$\boxed{\begin{array}{l}\dim H(x) \leq \dim H(y) \\ \text{whenever } x \in E_1 \text{ and } y \in E.\end{array}}$$

We will pay particular attention to the lowest stratum E_1.

More precisely, we proceed as follows.

STEP 0

Our goal is to find a section F of the bundle $\mathcal{H}$, with C^m-norm $\leq CM$.

$$\boxed{\text{If such an } F \text{ exists, then what can we say about } J_x(F) \text{ for a given } x \in E\ ?}$$

Obviously,

$J_x(F) \in H(x)$, but we can say more.

In the spirit of a previous lecture, we will define a convex set $\Gamma(x, k, CM) \subset H(x)$.

Any section F with norm $\leq CM$ must satisfy
$J_x(F) \in \Gamma(x, k, CM)$ for $x \in E$.

STEP 1

We produce a function

$F_1 \in C^m(\mathbb{R}^n)$

with norm $\leq CM$,

such that

$J_x(F_1) \subset \Gamma(x, k, CM)$

for all x in the

LOWEST STRATUM E_1.

STEP 2

We CORRECT F_1

away from E_1 to produce

a fn. $F \in C^m(\mathbb{R}^n)$

with norm $\leq CM$

s.t.

$J_x(F) \in H(x)$

for ALL $x \in E$, not just

all x in the lowest stratum.

QMT asserts precisely that such an F exists, so we are DONE.

That's the PLAN.

LET'S CARRY IT OUT.

WE START BY DEFINING THE STRATA

Recall, $\mathcal{P} = \left[\begin{array}{l}\text{VECTOR SPACE of all} \\ m^{\underline{th}} \text{ DEGREE polys} \\ \text{on } \mathbb{R}^n\end{array}\right]$.

Let $\overline{\mathcal{P}} = \left[\begin{array}{l}\text{VECTOR SPACE of all} \\ (m-1)^{\underline{st}} \text{ DEGREE} \\ \text{polys on } \mathbb{R}^n\end{array}\right]$

For each $x \in \mathbb{R}^n$, there's a natural linear map

$$\pi_x : \mathcal{P} \to \overline{\mathcal{P}}$$

defined by

$$\pi_x P = \left[\begin{array}{l}(m-1)^{\underline{rst}} \text{ DEGREE} \\ \text{TAYLOR POLY of } P \\ \text{at } x\end{array}\right].$$

To each $x \in E$,

we associate two integers,

$$\boxed{\begin{array}{l} d(x) = \dim H(x) \\ \quad \text{and} \\ \bar{d}(x) = \dim \pi_x H(x). \end{array}}$$

For any two integers $d, \bar{d}$,

WE SET

$$E(d,\bar{d}) = \{x \in E : d(x) = d,\ \bar{d}(x) = \bar{d}\}.$$

The STRATA of E

are the

NON-EMPTY $E(d,\bar{d})$.

The LOWEST STRATUM E_1

is the STRATUM $E(d, \bar{d})$,

for which

$\boxed{\text{(A) } d \text{ is minimized among all strata,}}$

and

$\boxed{\text{(B) } \bar{d} \text{ is minimized among all strata that satisfy (A).}}$

Observe :

- There are at most $C(m,n)$ strata
  (and there's at least one).

- $\dim H(x)$ and $\dim \pi_x H(x)$ are constant on each stratum

Because $\mathcal{H} = (H(x))_{x \in E}$ is Glaeser stable, one checks easily that

$$\boxed{\text{THE LOWEST STRATUM } E_1 \text{ IS COMPACT.}}$$

We've defined the strata.

NEXT, LET'S DEFINE

$\Gamma(x, k, M)$.

(The definition will look

familiar.)

Let $x_0 \in E$, $M > 0$.

Then $\Gamma(x_0, k, M)$ consists of all $P_0 \in H(x_0)$ s.t.

Given $x_1, \dots, x_k \in E$

there exist $P_1 \in H(x_1), \dots, P_k \in H(x_k)$

satisfying

$|\partial^\alpha P_i(x_i)| \le M$ for $|\alpha| \le m$, $i = 0, 1, \dots, k$

and

$|\partial^\alpha (P_i - P_j)(x_i)| \le M |x_i - x_j|^{m - |\alpha|}$

for $|\alpha| \le m$, $i, j = 0, 1, \dots, k$.

Thus, $\Gamma(x_0, k, M)$ is a (possibly empty) convex subset of $\mathcal{P}$.

$\boxed{\text{As promised, any section } F \text{ of } \mathcal{H} \text{ with } C^m \text{ norm} \le M \text{ satisfies } J_x(F) \in \Gamma(x, k, M) \text{ for all } x \in E.}$

(IMMEDIATE FROM TAYLOR'S THM.)

WE'VE DEFINED

THE STRATA,

THE LOWEST STRATUM,

and

THE $\Gamma(x, M, k)$,

and seen THEIR

BASIC PROPERTIES.

NOW, LET'S PRESENT THE PLAN of the PROOF of QMT in more DETAIL.

Plan of the Proof of QMT (Version 2.0)

We proceed by induction on the number of strata.

Fix $\Lambda \geq 1$, and assume the

$$\boxed{\begin{array}{l}\text{INDUCTION HYPOTHESIS:}\\ \text{QMT holds for all bundles}\\ \text{with fewer than } \Lambda \text{ strata}\end{array}}$$

(Holds vacuously if $\Lambda = 1$)

USING THE INDUCTION HYPOTHESIS, WE PROVE QMT for bundles with exactly 1 STRATA.

THAT WILL COMPLETE THE INDUCTION and prove QMT.

From now on, we fix

$\mathcal{H} = (H(x))_{x \in E}$ and $M > 0$

as in the hypotheses of QMT.

We suppose that $\mathcal{H}$ has exactly Λ strata.

Let $E_1 =$ lowest stratum.

OUR GOAL IS TO PRODUCE an $F \in C^m(\mathbb{R}^n)$ such that

$$\boxed{\begin{array}{l} J_x(F) \in H(x) \text{ for all } x \in E \\ \text{and} \\ \|F\|_{C^m(\mathbb{R}^n)} \leq CM. \end{array}}$$

To do so, we proceed as follows ...

STEP 1 :

WE PRODUCE A FN.

$F_1 \in C^m(\mathbb{R}^n)$

with norm $\leq CM$

such that

$$\boxed{\begin{array}{l} J_x(F_1) \in \Gamma(x, k, CM) \subset H(x) \\ \text{for all } x \text{ in the} \\ \text{LOWEST STRATUM } E_1 \end{array}}$$

STEP 2:

WE CORRECT F_1

AWAY FROM E_1, to

PRODUCE $F \in C^m(\mathbb{R}^n)$

WITH NORM $\leq CM$

SUCH THAT

$J_x(F) \in H(x)$ for all $x \in E$.

(THAT WAS OUR GOAL.)

That will complete our induction on Λ, thus proving QMT.

STEP 1 is the hard part, so let's deal with STEP 2 first.

SO WE SUPPOSE FOR THE MOMENT THAT WE ALREADY HAVE $F_1 \in C^m(\mathbb{R}^n)$ WITH NORM $\leq CM$, S.T. $J_x(F) \in \Gamma(x, k, CM)$ for $x \in E_1$.

OUR TASK IS TO CORRECT F_1 away from E_1.

WE START BY FOLLOWING WHITNEY'S CLASSICAL PROOF.

RECALL THAT E_1 IS COMPACT.

WE MAY THEREFORE PARTION $\mathbb{R}^n \setminus E_1$ INTO WHITNEY CUBES Q.

For each Whitney cube Q, we have $E_1 \cap Q^* = \emptyset$, but there is a point $x_Q \in E_1$ within distance $C\delta_Q$ of Q.

[Here & throughout, δ_Q = sidelength of Q.]

We introduce the Whitney Partition of Unity

$$\sum_Q \theta_Q = 1 \text{ on } \mathbb{R}^n \setminus E_1,$$

where

$$\operatorname{supp} \theta_Q \subset Q^*$$

and the usual estimates hold.

Our PLAN is to set

$$\boxed{F = F_1 + \sum_Q \theta_Q F_Q,}$$

where F_Q corrects F_1

locally in $\operatorname{supp} \theta_Q$.

More precisely, we

want to produce $F_Q \in C^m(\mathbb{R}^n)$

with the following properties:

DESIRED PROPERTIES OF THE F_Q

(A) $J_x(F_1 + F_Q) \in H(x)$

for all $x \in Q^* \cap E$.

(F_Q "CORRECTS" F_1 on Q^*)

(B)

$$|\partial^\alpha F_Q(x)| \leq C M \delta_Q^{m-|\alpha|}$$

for $x \in Q^*$, $|\alpha| \leq m$.

(c)

$$\frac{\sup_{x \in Q^*} |\partial^\alpha F_Q(x)|}{\delta_Q^{m-|\alpha|}} \to 0$$

as $\delta_Q \to 0$,

for $|\alpha| \leq m$.

Those are the properties we want for the F_Q.

First, we check that if the F_Q satisfy (A), (B), (C), then $F = F_1 + \sum_Q \theta_Q F_Q$ does what we want.

Then we show how to produce such F_Q, USING THE INDUCTION HYP.

So suppose the F_Q satisfy (A), (B), (C).

Using the estimates (B), (C), we can check that

$$\boxed{\sum_Q \theta_Q F_Q \in C^m(\mathbb{R}^n) \text{ with norm } \leq CM,}$$

and

$$\boxed{J_x\left(\sum_Q \theta_Q F_Q\right) = 0 \text{ for } x \in E_1.}$$

We are assuming that our F_1 satisfies

$$\boxed{\| F_1 \|_{C^m(\mathbb{R}^n)} \leq CM}$$

and

$$\boxed{\begin{array}{l} J_x(F_1) \in \Gamma(x, k, CM) \subset H(x) \\ \text{for } x \in E_1. \end{array}}$$

Combining our results for F_1 with those for $\sum_Q \theta_Q F_Q$, we find that

$$\boxed{\left\| F_1 + \sum_Q \theta_Q F_Q \right\|_{C^m(\mathbb{R}^n)} \leq CM}$$

and that

$$\boxed{J_x\left(F_1 + \sum_Q \theta_Q F_Q\right) = J_x(F_1) \in H(x) \text{ for } x \in E_1.}$$

We still must show that

$$\boxed{\begin{gathered} J_x\Big(F_1 + \sum_Q \theta_Q F_Q\Big) \in H(x) \\ \text{for } x \in E \setminus E_1. \end{gathered}}$$

To do so, we bring in property (A) of the F_Q.

Let $x \in E \setminus E_1$.

In a nbd of x, we have $\sum_Q \theta_Q = 1$,

hence $F_1 + \sum_Q \theta_Q F_Q = \sum_Q \theta_Q \cdot [F_1 + F_Q]$.

Consequently,

$$J_x\left(F_1 + \sum_Q \theta_Q F_Q\right) = \sum_Q J_x(\theta_Q) \odot J_x(F_1 + F_Q)$$

JET MULT.
AT x

PROPERTY (A) TELLS US THAT

$$J_x(F_1 + F_Q) \in H(x)$$

for each Q s.t. $x \in \operatorname{supp} \theta_Q$.

BECAUSE $H(x)$ IS A TRANSLATE OF AN IDEAL IN $\mathcal{R}_x$,

IT FOLLOWS THAT

$$\boxed{J_x\left(F_1 + \sum_Q \theta_Q F_Q\right) \in H(x)}$$

SO WE'VE PROVEN THAT

$$F = F_1 + \sum_Q \theta_Q F_Q$$

satisfies

- $\|F\|_{C^m} \leq CM$,

- $J_x(F) \in H(x)$ for $x \in E_1$

and

- $J_x(F) \in H(x)$ for $x \in E \setminus E_1$.

Thus, F has all the desired properties for

STEP 2 -

It corrects F_1, and it has C^m norm $\leq CM$.

WE'VE NOW SHOWN THAT

IF

WE CAN FIND FNS. $F_Q \in C^m(\mathbb{R}^n)$

WITH PROPERTIES (A), (B), (C),

THEN

WE CAN CORRECT the fn. F_1

produced by STEP 1,

& thus accomplish STEP 2.

WE NEXT EXPLAIN HOW TO PRODUCE FNS. F_Q THAT SATISFY (A), (B), (C).

THAT WILL COMPLETE OUR DISCUSSION OF STEP 2.

WE WILL THEN EXPLAIN HOW TO CARRY OUT STEP 1.

FINDING F_Q THAT SATISFY (A), (B), (C).

Let's first recall (A), (B), (C).

(A) $J_x(F_1 + F_Q) \in H(x)$ for $x \in Q^* \cap E$.

(B) $|\partial^\alpha (F_Q)| \leq C M \delta_Q^{m-|\alpha|}$ on Q^* for $|\alpha| \leq m$.

(C) $$\frac{\sup_{x \in Q^*} |\partial^\alpha F_Q|}{\delta_Q^{m-|\alpha|}} \longrightarrow 0$$

as $\delta_Q \to 0$ (for $|\alpha| \leq m$).

To produce F_Q satisfying (A), (B), (C),

we will look at the bundle

$$\mathcal{H}\Big|_{E \cap Q^*} = \big(H(x)\big)_{x \in E \cap Q^*}.$$

DETAIL: WE TAKE Q^* TO BE A CLOSED CUBE, BECAUSE $E \cap Q^*$ MUST BE COMPACT, AS REQUIRED BY THE DEF. OF A BUNDLE.

The KEY POINT IS THAT

$\mathcal{H}|_{E \cap Q^*}$ has $< \Lambda$ STRATA,

because Q^* DOESN'T INTERSECT THE LOWEST STRATUM E_1.

Therefore, the INDUCTION HYP. applies.

Immediately from the

INDUCTION HYP.,

we obtain fns F_Q

that satisfy (A).

To obtain (A), (B), (C),

we have to be more careful.

WE WILL RECENTER and RESCALE

$$\mathcal{H}\big|_{E \cap Q^*}$$

in order to obtain F_Q

that satisfy (A) and (B)

We'll worry about (C) later.

WE RECENTER $\mathcal{H}|_{E \cap Q^*}$

by setting

$$\boxed{\begin{gathered} \mathcal{H}_Q^\# = \left(H_Q^\#(x) \right)_{x \in E \cap Q^*}, \\ \text{where} \\ H_Q^\#(x) = \left\{ P - J_x(F_1) : P \in H(x) \right\}. \end{gathered}}$$

A fn. F is a SECTION OF $\mathcal{H}_Q^\#$

$\iff$ $F_1 + F$ is a SECTION OF $\mathcal{H}|_{E \cap Q^*}$.

NEXT, WE **RESCALE** $\mathcal{H}^{\#}_Q$.

Let $\phi_Q : \mathbb{R}^n \to \mathbb{R}^n$ be the affine map that sends the unit cube Q° to Q^*.

$$\boxed{\begin{array}{l}\text{Then set } \hat{\mathcal{H}}_Q = \left(\hat{H}_Q(\hat{x})\right)_{\hat{x} \in \hat{E}_Q} \\ \text{where } \hat{E}_Q = \phi_Q^{-1}(E \cap Q^*) \\ \text{and} \\ \hat{H}_Q(\hat{x}) = \left\{ P \circ \phi_Q : P \in H^{\#}_Q(\phi_Q(\hat{x})) \right\}\end{array}}$$

JUST LIKE $\mathcal{H}|_{E \cap Q^*}$,

the bundle $\hat{\mathcal{H}}_Q$

has $< \Lambda$ STRATA,

so the INDUCTION HYP.

applies.

RECALL THAT $x_Q \in E_1$

lies within distance $C\delta_Q$ of Q.

Also,

$\|F_1\|_{C^m} \leq CM$

and

$J_{x_Q}(F_1) \in \Gamma(x_Q, k, CM)$

from the (as yet unfulfilled)

STEP 1.

These facts yield the estimate

$$\boxed{\|\hat{\mathcal{H}}_Q\| \leq C M \delta_Q^m,}$$

once we recall the definitions of

$\hat{\mathcal{H}}_Q$, $\Gamma(\ldots)$, $\|\cdot\|$ for bundles.

(TRUST ME!)

Applying the INDUCTION HYP.

(QMT for bundles with $< \Lambda$ STRATA),

we obtain a section

$\hat{F}_Q$ of $\hat{\mathcal{H}}_Q$, with

C^m norm at most $C M \delta_Q^m$

Undoing the effect of the rescaling by ϕ_Q, we obtain a function $F_Q \in C^m(\mathbb{R}^n)$ that satisfies

$$\boxed{\text{(A)} \quad J_x(F_1 + F_Q) \in H(x) \text{ for } x \in E \cap Q^*}$$

and

$$\boxed{\text{(B)} \quad |\partial^\alpha F_Q| \le C M \delta_Q^{m-|\alpha|} \text{ on } \mathbb{R}^n.}$$

So, exploiting the INDUCTION HYP,

we have produced fns F_Q

that satisfy (A) and (B).

We have made CRUCIAL USE

of the property

$J_x(F_1) \in \Gamma(x, k, CM)$

from STEP 1, to bound $\|\hat{\mathcal{H}}_Q\|$.

WE STILL HAVE TO COME

TO GRIPS WITH (C).

THERE ISN'T TIME TO

EXPLAIN FULLY, BUT

HERE ARE A FEW REMARKS.

RECALL THAT OUR BUNDLE $\mathcal{H}$

IS GLAESER STABLE.

That means the following ....

Let $x_0 \in E$ and $P_0 \in H(x_0)$.

Given $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$\boxed{\begin{array}{l}\text{Given } x_1, \dots, x_k \in E \cap B(x_0, \delta) \\ \text{there exist } P_1 \in H(x_1), \dots, P_k \in H(x_k) \\ \text{such that} \\ |\partial^\alpha (P_i - P_j)(x_i)| \le \varepsilon |x_i - x_j|^{m-|\alpha|} \\ \text{for } |\alpha| \le m, \quad i,j = 0,1,\dots,k.\end{array}}$$

THE MAIN POINT IN ACHIEVING (C) IS TO SHOW THAT THE ABOVE δ CAN BE CHOSEN INDEPENDENTLY OF x_0, P_0, PROVIDED

$$\boxed{x_0 \in E_1 \ \& \ |\partial^\alpha P_0(x_0)| \text{ BOUNDED.}}$$

The precise formulation

of this

GOOD NEWS

is as follows.

LEMMA: Given $\varepsilon, M' > 0$, $\exists\, \delta > 0$ for which the following holds:

Let $x_0, x_1, \cdots, x_k \in E$ and $P_0 \in H(x_0)$.

Suppose $x_0 \in E_1$, $x_1, \cdots, x_k \in B(x_0, \delta)$ and $|\partial^\alpha P_0(x_0)| \leq M'$ for $|\alpha| \leq m$.

Then $\exists\, P_1 \in H(x_1), \cdots, P_k \in H(x_k)$ s.t.

$$|\partial^\alpha (P_i - P_j)(x_i)| \leq \varepsilon |x_i - x_j|^{m-|\alpha|}$$

for $i, j = 0, 1, \cdots, k$ and $|\alpha| \leq m$.

The LEMMA is roughly analogous to the elementary fact that a continuous fn. on a compact set is uniformly continuous.

To prove the LEMMA,

WE USE

- COMPACTNESS of E_1.
- CONSTANCY OF $\dim H(x)$ on E_1
- HELLY'S THM. (k CHANGES!)

LET'S JUST DECLARE VICTORY,

&

BELIEVE THAT WE CAN PRODUCE

F_Q SATISFYING (A), (B), (C).

For future reference, we

note a simple Corollary of

the above LEMMA.

Corollary:

Given $\varepsilon, M' > 0$ $\exists \delta > 0$ s.t.:

Let $x \in E_1$, $P \in H(x)$, $y \in E_1 \cap B(x, \delta)$.

Suppose $|\partial^\alpha P(x)| \leq M'$ for $|\alpha| \leq m$.

Then there exists $P' \in H(y)$ s.t.

$$|\partial^\alpha (P - P')(x)| \leq \varepsilon M' |x - y|^{m - |\alpha|}$$

for $|\alpha| \leq m$.

HERE'S WHERE WE STAND

PLAN OF PROOF OF QMT

Fix $\Lambda \geq 1$, & assume QMT for bundles with $< \Lambda$ STRATA.

Fix $\mathcal{H} = (H(x))_{x \in E}$ & $M > 0$ as in hyp. of QMT.

Assume $\mathcal{H}$ has exactly Λ strata.

$\boxed{\text{STEP 1: Produce } F_1 \in C^m(\mathbb{R}^n) \text{ with norm} \leq CM, \text{ s.t.}}$ ← IOU

$J_x(F) \in \Gamma(x, k, CM)$ for all $x \in E_1$

STEP 2: Correct F_1 to prove QMT.

To carry out STEP 2,

we produced fns F_Q that

correct F_1 locally on Q^*

for each Whitney cube Q.

We then patched together the F_Q

to produce a fn

$$F = F_1 + \sum_Q \theta_Q F$$

that does what we want.

All that's missing from our proof of QMT is

STEP 1.

That's the hard part.

SO WE MUST PRODUCE

$$F_1 \in C^m(\mathbb{R}^n)$$

such that

$$J_x(F_1) \in \Gamma(x, k, CM)$$

for all $x \in E_1$

and

$$\|F_1\|_{C^m(\mathbb{R}^n)} \leq CM.$$

THAT WILL COMPLETE THE PROOF OF QMT

To produce such an F_1,

we bring in the

FINITENESS THM

from a previous lecture,

in its MOST GENERAL FORM*

* (SO FAR)

RECALL THE FINITENESS THM:

WE WORK IN $C^{m,\omega}(\mathbb{R}^n)$,

WHERE ω IS A MODULUS OF CONTINUITY,

i.e.

$\omega : [0,\infty) \longrightarrow [0,1]$

$\omega(0) = \lim_{t \to 0^+} \omega(t) = 0$

$\omega(t) = 1$ for $t \geq 1$

$\omega(t)$ is INCREASING on $[0,\infty)$

$\omega(t)/t$ is DECREASING on $(0,\infty)$

RECALL THAT $F \in C^{m,\omega}(\mathbb{R}^n)$ if

F and its derivatives up to order m are continuous, and for some $M < \infty$ we have

$$|\partial^\alpha F(x)| \leq M \text{ for } |\alpha| \leq m, \; x \in \mathbb{R}^n$$

and

$$|\partial^\alpha F(x) - \partial^\alpha F(y)| \leq M\, \omega(|x-y|)$$

for $|\alpha| = m$, $x, y \in \mathbb{R}^n$.

The least such M is the $C^{m,\omega}(\mathbb{R}^n)$ norm of F.

Now suppose $E_1 \subset \mathbb{R}^n$ is an (arbitrary) set.

For each $x \in E_1$, suppose we are given an m^{th} degree poly. (an "m-jet") $f(x) \in \mathcal{P}$, and a symmetric convex set $\sigma(x) \subset \mathcal{P}$.

Let $M > 0$ be given.

We want to find a function $F_1 \in C^{m,\omega}(\mathbb{R}^n)$ such that

$$\boxed{\begin{gathered} J_x(F_1) \in f(x) + M\sigma(x) \\ \text{for all } x \in E_1, \\ \text{and} \\ \|F_1\|_{C^{m,\omega}(\mathbb{R}^n)} \leq M. \end{gathered}}$$

There is a FINITENESS THM

for such problems,

provided the

SYMMETRIC CONVEX SETS

$\sigma(x) \subset \mathcal{P}$

ARE "WHITNEY ω-CONVEX".

DEFINITION :

Suppose we are given

- $\sigma \subset \mathcal{P}$, a symmetric convex set
- ω, a modulus of continuity
- $x_0 \in \mathbb{R}^n$, a point,

  and
- $C_w \geq 1$, a constant.

Then we say that σ is

WHITNEY ω-CONVEX AT x_0

WITH WHITNEY CONSTANT C_w,

if the following holds ...

DEF. OF WHITNEY ω-CONVEXITY (CONTINUED)

Let $P \in \sigma$, $Q \in \mathcal{P}$, $0 < \delta \leq 1$.

Suppose that

$$|\partial^\alpha P(x_0)| \leq \delta^{m-|\alpha|}\,\omega(\delta)$$

and

$$|\partial^\alpha Q(x_0)| \leq \delta^{-|\alpha|}$$

for $|\alpha| \leq m$.

Then $Q \odot_{x_0} P \equiv J_{x_0}(QP)$ belongs to $C_w \sigma$.

Similarly, σ is called

WHITNEY CONVEX AT x_0

WITH WHITNEY CONSTANT C_W

if the following holds ...

DEF. OF WHITNEY CONVEXITY (CONTINUED)

Let $P \in \sigma$, $Q \in \mathcal{P}$, $0 < \delta \leq 1$.

Suppose that

$$|\partial^\alpha P(x_0)| \leq \delta^{m-|\alpha|}$$

NOTE: NO FACTOR $\omega(\delta)$.

and

$$|\partial^\alpha Q(x_0)| \leq \delta^{-|\alpha|}$$

for $|\alpha| \leq m$.

Then $Q \odot_{x_0} P \equiv J_{x_0}(QP)$

belongs to $C_w \sigma$.

Trivially,

$$\boxed{\begin{array}{l}\sigma \text{ Whitney convex at } x_0 \\ \text{with Whitney const. } C_W\end{array}}$$

$$\Downarrow$$

$$\boxed{\begin{array}{l}\sigma \text{ Whitney } \omega\text{-convex at } x_0 \\ \text{with Whitney const. } C_W\end{array}}$$

for any modulus of continuity ω.

GIVEN THE ABOVE BACKGROUND

WE CAN NOW STATE THE

FINITENESS THM for $C^{m,\omega}$

Thm: Given $m, n \geq 1$, there exists $k = k(m,n)$ for which the following holds.

Let ω be a modulus of continuity.

Let $E_1 \subset \mathbb{R}^n$ be an (arbitrary) set.

For each $x \in E_1$, let $f(x) \in \mathcal{P}$ and $\sigma(x) \subset \mathcal{P}$ be given.

Let $C_w > 0$ be given.

Suppose that for each $x \in E_1$, the set $\sigma(x)$ is Whitney ω-convex at x with Whitney constant C_w.

Let $M > 0$.

Suppose that given any $x_1, \dots, x_k \in E$

there exist

$$P_1 \in f(x_1) + M\sigma(x_1), \dots, P_k \in f(x_k) + M\sigma(x_k)$$

such that

$$|\partial^\alpha P_i(x_i)| \leq M \quad \text{for } |\alpha| \leq m, \; i = 1, \dots, k$$

and

$$|\partial^\alpha (P_i - P_j)(x_i)| \leq M\omega(|x_i - x_j|) \cdot |x_i - x_j|^{m - |\alpha|}$$

for $|\alpha| \leq m$, $i, j = 1, \dots, k$.

THE "FINITENESS CONDITION"

Then there exists

$F_1 \in C^{m,\omega}(\mathbb{R}^n)$

such that

$$\boxed{\begin{array}{l} J_x(F_1) \in f(x) + CM\sigma(x) \\ \quad \text{for all } x \in E_1, \\ \\ \text{and} \\ \|F_1\|_{C^{m,\omega}(\mathbb{R}^n)} \leq CM. \end{array}}$$

Here, C DEPENDS ONLY on m, n and on the Whitney constant C_W.

REMARKS :

The above FINITENESS THM differs slightly from the version given in a previous lecture.

Let's see the differences, and check that they don't matter.

DISCREPANCY 1 :

In place of today's hypothesis

"Given $x_1, \dots, x_k \in E$, there exist

$P_1 \in f(x_1) + M\sigma(x_1), \dots, P_k \in f(x_k) + M\sigma(x_k)$

[satisfying ESTIMATES]"

the earlier FINITENESS THM

had a different-looking hypothesis

namely ...

For each subset $S \subset E_1$

with $\#(S) \le k$,

there exists $F^S \in C^{m,\omega}(\mathbb{R}^n)$

such that

$\|F^S\|_{C^{m,\omega}(\mathbb{R}^n)} \le M$

and

$J_x(F^S) \in f(x) + M\sigma(x)$

for all $x \in S$.

Today's hypothesis

and the hypothesis just quoted

from a previous lecture are

easily seen to be equivalent,

thanks to Whitney's extension

thm for $C^{m,\omega}(\mathbb{R}^n)$

(which is due to Glaeser)

P.S. In passing from one hypothesis to the other, M may be degraded to CM, where C depends only on m, n.

That doesn't hurt.

SO MUCH FOR DISCREPANCY 1.

DISCREPANCY 2 :

In the previous lecture, we assumed that our set $E_1 \subset \mathbb{R}^n$ is FINITE.

Today we take E_1 ARBITRARY.

However, the FINITENESS THM for general E_1 follows EASILY from the case of arbitrary finite E_1 by a compactness argument involving Ascoli's THM.

(That's because we are dealing with $C^{m,\omega}$, NOT C^m.)

That takes care of the SECOND DISCREPANCY.

There are no further discrepancies between the FINITENESS THM stated today & that discussed in an earlier lecture.

SO WE TAKE TODAY'S

FINITENESS THM

as

KNOWN.

WHERE DO WE STAND?

We are given a bundle

$\mathcal{H} = (H(x))_{x \in E}$

(Glaeser stable, non-empty fibers)

with lowest stratum E_1.

We must produce $F \in C^m(\mathbb{R}^n)$

with norm $\leq CM$ such that

$J_x(F) \in \Gamma(x, k, CM)$

for all $x \in E_1$.

To do so, we hope to use a

FINITENESS THM.

that tells us when we can find

$F \in C^{m,\omega}(\mathbb{R}^n)$

with norm $\leq CM$,

such that

$J_x(F) \in f(x) + CM\sigma(x)$

for all $x \in E_1$.

JUST AS IN A PREVIOUS LECTURE,

the $\Gamma(x, k, M)$ (ESSENTIALLY)

have the form

$$\Gamma(x, k, M) \approx f(x) + M\sigma(x),$$

for suitable f, σ.

More precisely,

$$\boxed{\Gamma(x, k, M) \subset f(x) + 2M\sigma(x) \subset \Gamma(x, k, 3M)}$$

Furthermore, the $\sigma(x)$

are Whitney convex

with Whitney constant

depending only on m, n.

Hence, the $\sigma(x)$ are

Whitney ω-convex

(with the same Whitney const.)

for ANY ω.

So,

WE ARE TRYING TO APPLY THE FINITENESS THM for $C^{m,\omega}$.

We know that our $\Gamma(x, k, M)$ are as assumed in the FINITENESS THM

but

We haven't yet defined an ω

and

We haven't yet checked the FINITENESS CONDITION.

Therefore, to complete the proof of QMT, we must meet the following

CHALLENGE ....

CHALLENGE :

FIND A MODULUS OF CONTINUITY ω for which the following holds –

Given $x_1, \cdots, x_k \in E_1$

there exist

$P_1 \in \Gamma(x_1, k', CM), \cdots, P_k \in \Gamma(x_k, k', CM)$

s.t.

$|\partial^\alpha P_i(x_i)| \leq CM$ for $|\alpha| \leq m$, $i = 1, \cdots, k$

and

$$|\partial^\alpha (P_i - P_j)(x_i)| \leq CM\,\omega(|x_i - x_j|) \cdot |x_i - x_j|^{m-|\alpha|}$$

for $|\alpha| \leq m$, $i, j = 1, \cdots, k$.

WE WILL MEET THE

CHALLENGE,

thus proving QMT.

TOOLS TO MEET THE CHALLENGE

WE WILL USE 3 TOOLS —

ONE FOR COARSE LENGTSCALES,

ONE FOR FINE LENGTHSCALES,

and

ONE TO COMBINE RESULTS FROM DIFFERENT LENGTHSCALES

TOOL I (COARSE LENGTHSCALES)

RECALL THE HYPOTHESIS

$\|\mathcal{H}\| \leq M$ for QMT.

By definition, that's the following assertion...

Given $x_1, \dots, x_k \in E$

there exist

$P_1 \in H(x_1), \dots, P_k \in H(x_k)$

such that

$|\partial^\alpha P_i(x_i)| \leq M$ for $|\alpha| \leq m$, $i = 1, \dots, k$

and

$|\partial^\alpha (P_i - P_j)(x_i)| \leq M |x_i - x_j|^{m - |\alpha|}$

for $|\alpha| \leq m$, $i, j = 1, \dots, k$.

RECALLING THE DEFINITION

OF $\Gamma(x, k', M)$,

AND APPLYING HELLY'S THM

IN THE USUAL WAY,

WE OBTAIN THE FOLLOWING

RESULT ...

Suppose $k \gg k_0 \gg 1$

and $k' \gg k_0' \gg 1$.

Given $x_1, \dots, x_{k_0} \in E$, there exist

$P_1 \in \Gamma(x_1, k_0', M), \dots, P_{k_0} \in \Gamma(x_{k_0}, k_0', M)$

such that

$$|\partial^\alpha P_i(x_i)| \le CM$$

for $|\alpha| \le m$, $i = 1, \dots, k_0$

and

$$|\partial^\alpha (P_i - P_j)(x_i)| \le CM |x_i - x_j|^{m-|\alpha|}$$

for $|\alpha| \le m$, $i, j = 1, \dots, k_0$.

$$\boxed{\text{That's just what we need, except for a missing factor } \omega(|x_i - x_j|).}$$

We will take $\omega(t) \equiv 1$

for $t \geq \tau_0$

(τ_0 IS SOME VERY SMALL NUMBER TO BE DETERMINED.)

Then the above result gives us what we want, provided $x_1, \ldots, x_{k_0}$ contain no "VERY CLOSE NEIGHBORS",

i.e.

$|x_i - x_j| \geq \tau_0$ for $i \neq j$.

Note: WE HAVE HERE k_0, k_0'

INSTEAD OF k, k'.

That's OK.

We take k_0', k_0 as large

as we need, then make sure

that $k \gg k_0$

and $k' \gg k_0'$.

In what follows, let's ignore

the difference between k, k' & k_0, k_0'.

That's TOOL 1.

It deals with COARSE LENGTHSCALES

(all $|x_i - x_j| \geq \tau_0$, $i \neq j$)

TOOL II (FINE LENGTHSCALES)

WE RECALL SOME

GOOD NEWS

FROM OUR EARLIER

DISCUSSION:

Given $\varepsilon > 0$, $M' > 0$, $\exists \delta > 0$ s.t.:

$$\boxed{\begin{array}{l}\text{Let } x \in E_1,\ P \in H(x),\ y \in E_1 \cap B(x,\delta). \\ \text{Suppose } |\partial^\alpha P(x)| \leq M' \text{ for } |\alpha| \leq m. \\ \text{Then there exists } P' \in H(y) \text{ s.t.} \\ |\partial^\alpha (P - P')(x)| \leq \varepsilon M' |x-y|^{m-|\alpha|} \\ \text{for } |\alpha| \leq m.\end{array}}$$

We will use this fact

(with $M' = [\text{LARGE CONSTANT}] \cdot M$)

for $\varepsilon = \varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots,$

where $\varepsilon_\ell = 2^{-\ell}$ (EACH ℓ).

We obtain

$\delta = \delta_0, \delta_1, \delta_2, \cdots$

satisfying the conditions on the previous slide.

SMALLER $\delta \Rightarrow \left[\begin{array}{l}\text{RESULT GETS} \\ \text{WEAKER}\end{array}\right]$.

So, without loss of generality, we may pick the δ_ℓ so that

$\delta_0 < \hat{\delta}_0 \leftarrow$ A SMALL POS. NO. TO BE PICKED LATER

and

$\delta_{\ell+1} \leq \delta_\ell / 1000 \leftarrow$ FIRST PICK δ_0, THEN δ_1, NEXT δ_2, ...

DEFINE A MODULUS OF CONTINUITY

$\omega(t) = 1$ for $t \geq \delta_0$

$\omega(\delta_\ell) = \varepsilon_\ell$ for $\ell = 0, 1, 2, \dots$

$\omega(t)$ is linear for $t \in [\delta_{\ell+1}, \delta_\ell]$

for each $\ell = 0, 1, 2, \dots$

$\omega(0) = 0.$

We check easily that the above $\omega(t)$ is a MODULUS of continuity, thanks to the RAPID DECREASE of the δ_ℓ.

Our GOOD NEWS can be reformulated in terms of the modulus of continuity ω.

Let $x \in E_1$, $y \in E_1 \cap B(x, \delta_0)$.

Let $P \in H(x)$, with $|\partial^\alpha P(x)| \le CM$ (all $|\alpha| \le m$).

Then there exists $P' \in H(y)$ s.t.

$$|\partial^\alpha (P - P')(x)| \le CM\, \omega(|x-y|)\, |x-y|^{m-|\alpha|}$$

(all $|\alpha| \le m$).

AHA! We've found a modulus of continuity ω, and we have the desired factor $\omega(|x-y|)$ in our estimates.

That's TOOL II.

It applies at FINE LENGTHSCALES, i.e., when $|x-y| \leq \delta_0$.

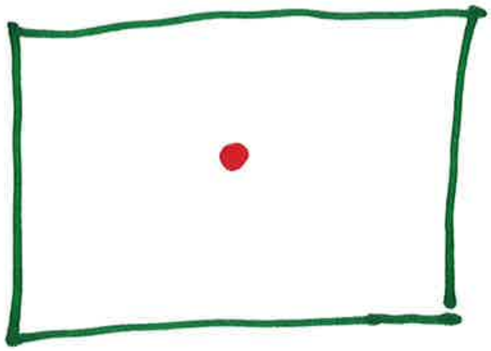
TOOL III :

VIEWING A k-POINT SET AT SEVERAL LENGTHSCALES.

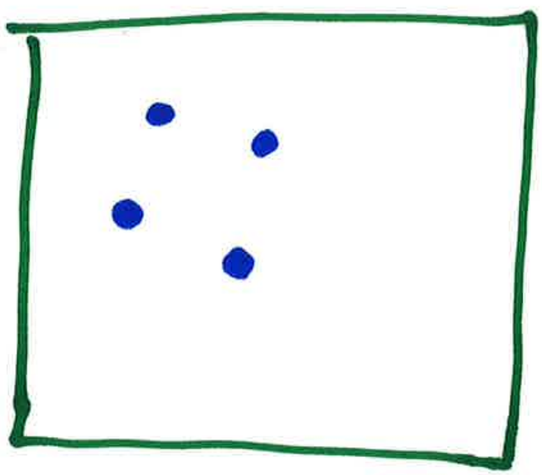
Let $S = \{x_1, \dots, x_k\} \subset \mathbb{R}^n$.

$$\boxed{\text{If we view } S \text{ from very far away, it looks to us like a single point.}}$$

$$\boxed{\text{As we approach } S\text{, what appeared to us as a single point now looks like a CLUSTER}}$$



S VIEWED FROM
FAR AWAY

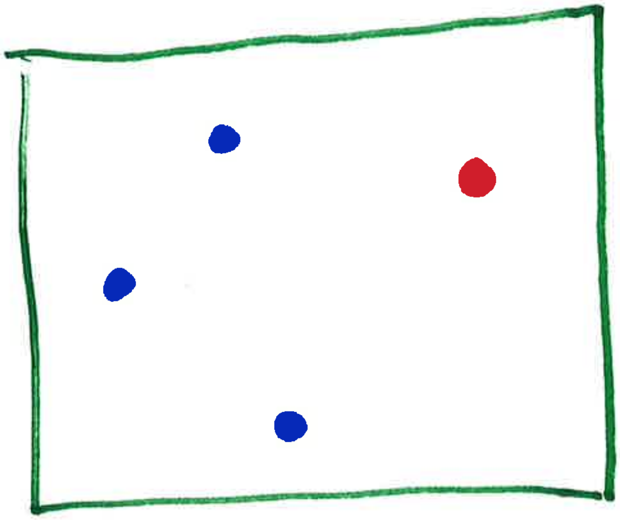


S VIEWED FROM
NOT SO
FAR AWAY

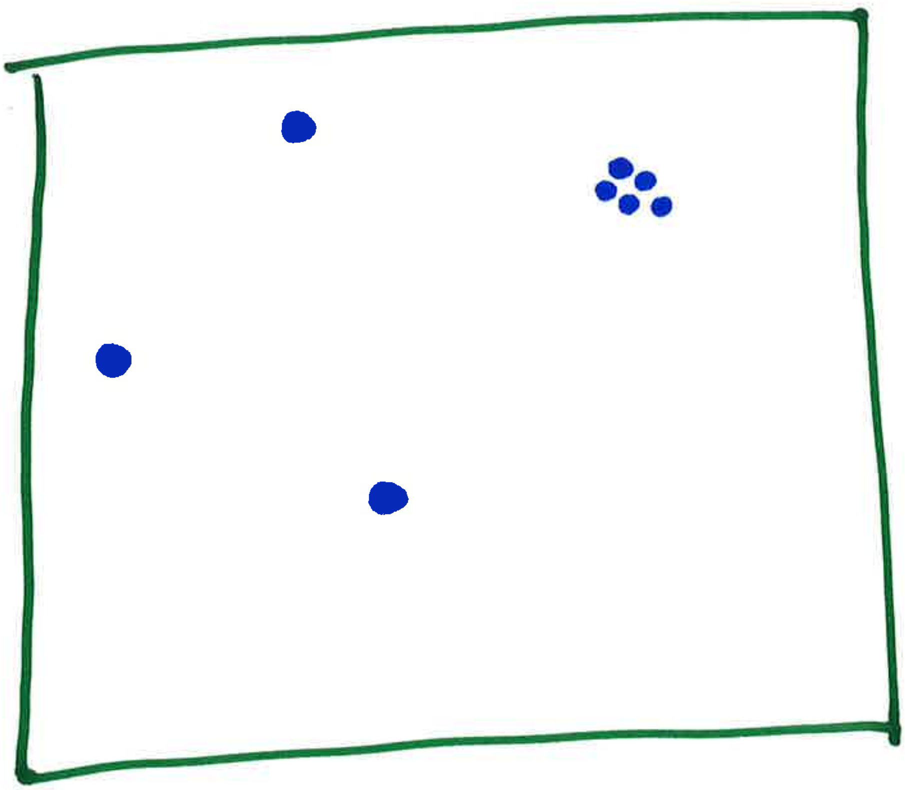
All distances between points in the cluster are comparable.

(If two lengthscales in the cluster were very different, we would notice the larger one before we perceive the smaller.)

As we approach S still more closely, what appeared to us before as a single point of the cluster BREAKS UP into a "subcluster".



S VIEWED FROM DISTANCE D_1



S VIEWED FROM A DISTANCE $<< D_1$

WITHIN THE SUBCLUSTER,

ALL DISTANCES ARE

COMPARABLE,

for the same reason as before.

PERHAPS SEVERAL

POINTS OF THE CLUSTER

RESOLVE INTO SUBCLUSTERS

AT THE SAME LENGTHSCALE.

That's OK.

As we approach still closer, our SUBCLUSTERS will resolve into sub-sub-clusters, and so on...

LET'S MAKE THESE IDEAS PRECISE.

Let $S \subset \mathbb{R}^n$ be finite.

Let $\hat{S} \subset \mathbb{R}^n$ be a finite set containing a point $\hat{x} \in S$.

Suppose that

$$\boxed{\begin{array}{l} |x-y| \geq c \cdot \mathrm{diam}(\hat{S}) \\ \text{for all } x, y \in \hat{S} \text{ distinct,} \end{array}}$$

and that

$$\boxed{\begin{array}{l} \mathrm{diam}(\hat{S}) < \frac{1}{10} |x-y| \\ \text{for all } x, y \in S \text{ distinct.} \end{array}}$$

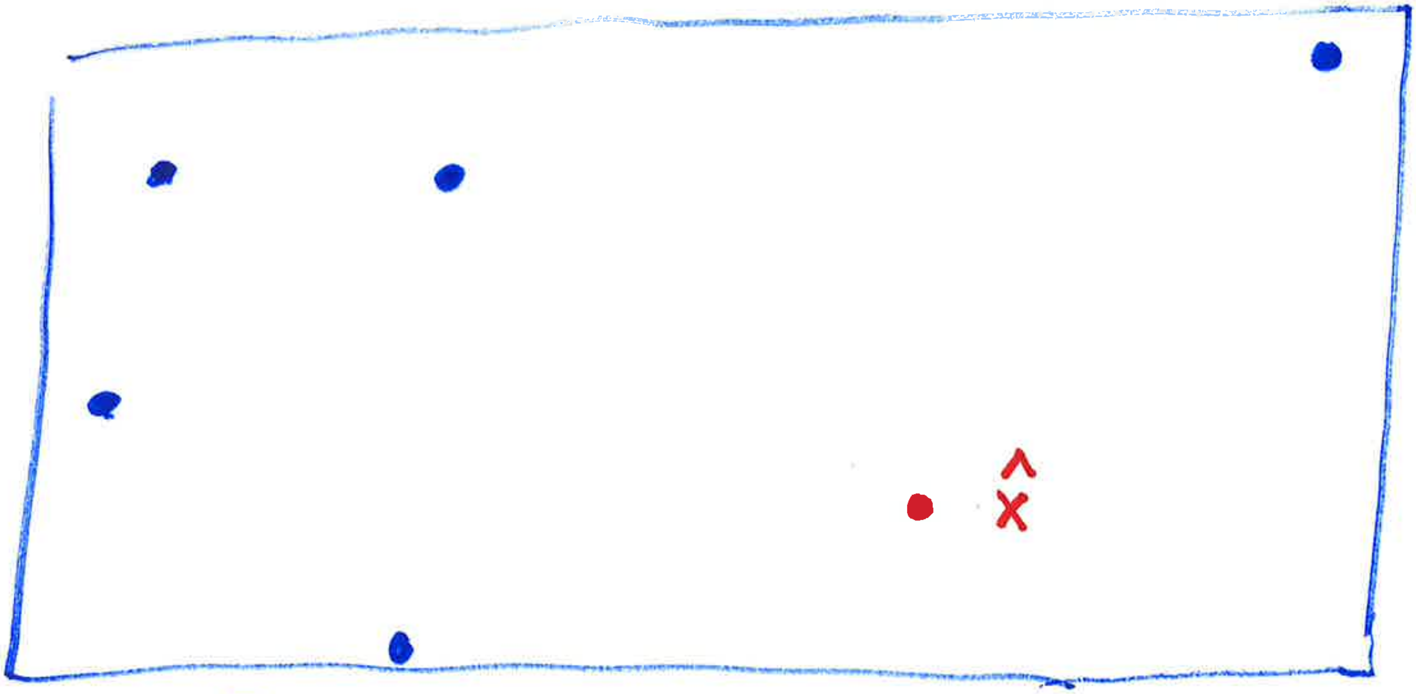
Then we say that

$S \cup \hat{S}$ arises from S

by SPROUTING $\hat{x}$ into $\hat{S}$.

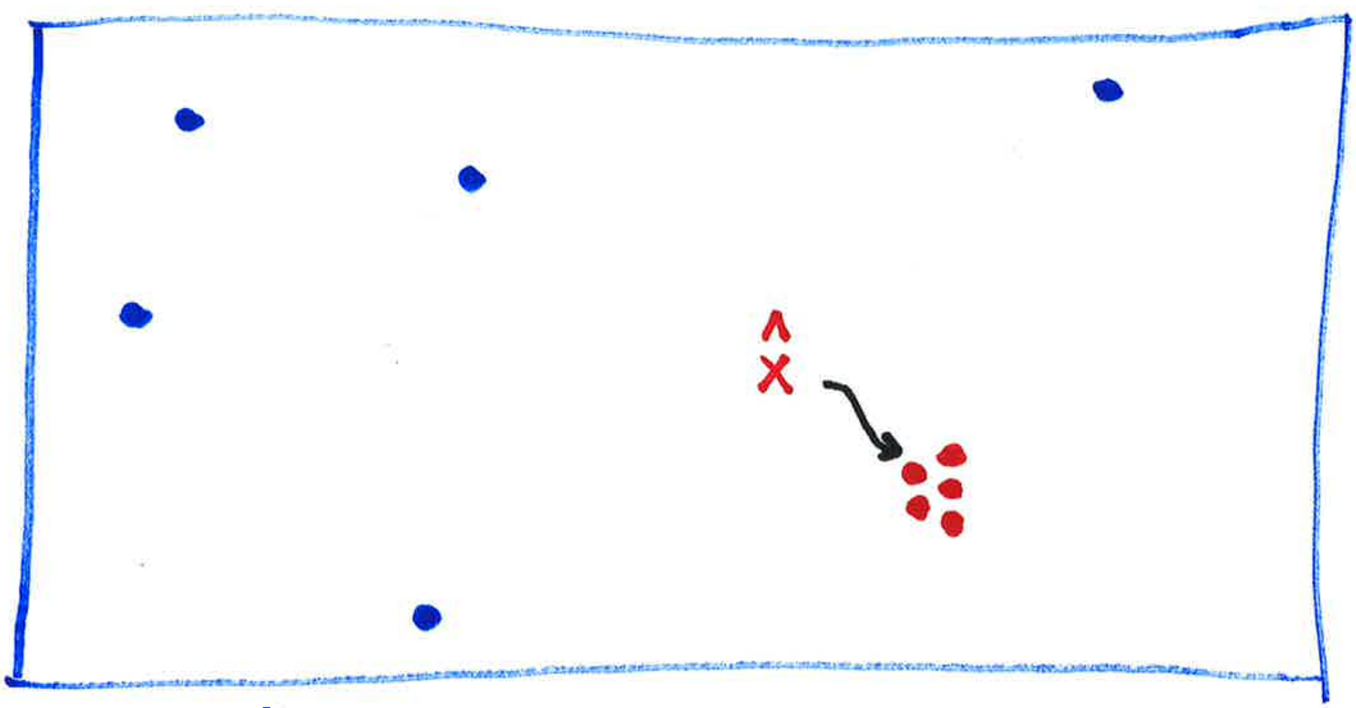
We call c the SPROUTING CONST.

$|x-y| \geq c \cdot \operatorname{diam} \hat{S}$ for $x, y \in \hat{S}$ distinct



S (includes $\hat{x}$)

BEFORE SPROUTING



$S \cup \hat{S}$ AFTER SPROUTING

($\hat{S}$ SHOWN IN RED)

SPROUTING LEMMA:

Let $S \subset \mathbb{R}^n$ be a k-point set, and let $\delta_0 > 0$.

Then there exist subsets

$$S_0 \subset S_1 \subset \cdots \subset S_{\nu_{max}} = S$$

with the following properties.

$$\boxed{\nu_{max} \leq k}$$

$$\boxed{|x-y| \gtrsim c\,\delta_0 \text{ for } x,y \in S_0 \text{ distinct.}}$$

For each $\nu < \nu_{max}$,
$S_{\nu+1}$ arises by sprouting some $x_\nu \in S_\nu$ into a set $\hat{S}_\nu$ of diameter $< \delta_0$.
Moreover, the sprouting const is $\gtrsim C$.

Here, c depends only on k, the number of points of S.

The easy proof of the SPROUTING LEMMA uses two observations.

$$\boxed{\text{There are at most } \binom{k}{2} \text{ distinct distances between distinct points of a } k\text{-point set } E.}$$

and ...

If for some $\tau > 0$,

no two $x, y \in S$ satisfy

$\tau \le |x-y| \le 2\tau$,

then the relation

$|x-y| \le \tau$

is an EQUIVALENCE RELATION

on S.

The Sprouting Lemma

is our Tool III.

We now have all three Tools

("Coarse scales", "Fine scales",

"Sprouting Lemma").

Using these Tools, we

will now meet our

Challenge.

RECALL, OUR

CHALLENGE

IS TO PROVE

THE FOLLOWING.

ONCE WE PROVE IT,
WE WILL HAVE PROVEN

QMT

CHALLENGE:

Given $x_1, \dots, x_k \in E_1$, produce

$P_1 \in \Gamma(x_1, k', CM), \dots, P_k \in \Gamma(x_k, k', CM)$

such that

$$|\partial^\alpha P_i(x_i)| \le CM \quad (|\alpha| \le m, \ i = 1, \dots, k)$$

and

$$|\partial^\alpha (P_i - P_j)(x_i)| \le CM\, \omega(|x_i - x_j|) \cdot |x_i - x_j|^{m-|\alpha|}$$

for $|\alpha| \le m$, $i, j = 1, \dots, k$.

To meet the CHALLENGE,

we write $S = \{x_1, \dots, x_k\} \subset E_1$

as in the SPROUTING LEMMA,

taking δ_0 in that lemma

to be as in TOOL II.

$\mathcal{S}_0$ $\boxed{\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{\nu_{MAX}} = \mathcal{S}.}$

$\boxed{|x-y| > c\delta_0 \text{ for } x, y \in \mathcal{S}_0 \text{ distinct}}$

$\mathcal{S}_{\nu+1}$ arises from $\mathcal{S}_\nu$ by SPROUTING x_ν into $\hat{\mathcal{S}}_\nu$

$\boxed{\text{diam } \hat{\mathcal{S}}_\nu < \delta_0 .}$

To each $x \in S$, we must associate

a poly. $P^x \in \Gamma(x, k', CM)$

s.t.

$$|\partial^\alpha P^x(x)| \leq CM \text{ for } |\alpha| \leq m, \ x \in S$$

and

$$|\partial^\alpha (P^x - P^y)(x)| \leq CM\,\omega(|x-y|) \cdot |x-y|^{m-|\alpha|}$$

for $|\alpha| \leq m$, $x, y \in S$.

PLAN:

We assign P^x to the pts $x \in S_0$,

then assign P^x to the pts $x \in S_1 \setminus S_0$,

next assign P^x to the pts $x \in S_2 \setminus S_1$,

$\vdots$

UNTIL FINALLY WE HAVE ASSIGNED P^x to all the pts

$x \in S_{\nu_{MAX}} = S$.

IMPLEMENTING THE PLAN FOR S_0

WE HAVE $|x-y| \geq c\delta_0$

for $x, y \in S_0$ distinct.

Therefore $\omega(|x-y|) \geq c$

for such x, y.

RECALL: $\omega(\delta_0) = 1$ and $\frac{\omega(t)}{t}$ is DECREASING.

TOOL I provides a family of polys $(P^x)_{x \in S_0}$

s.t.

$P^x \in \Gamma(x, k', CM)$ for each $x \in S_0$,

$|\partial^\alpha P^x(x)| \leq CM$ (all $|\alpha| \leq m$, $x \in S_0$)

and

$$|\partial^\alpha (P^x - P^y)(x)| \leq CM \, |x-y|^{m-|\alpha|}$$

for $|\alpha| \leq m$, $x, y \in S_0$.

Because $\omega(|x-y|) \gtrsim c$ for $x, y \in S_0$ distinct,

it follows trivially that

$$P^x \in \Gamma(x, k^\#, CM) \text{ for } x \in S_0$$

$$|\partial^\alpha P^x(x)| \leq CM \text{ for } |\alpha| \leq m, \; x \in S_0$$

and

$$|\partial^\alpha (P^x - P^y)(x)| \leq CM \, \omega(|x-y|) |x-y|^{m-|\alpha|}$$

for $|\alpha| \leq m$, $x, y \in S_0$

THAT'S JUST WHAT WE PROMISED!

So we have successfully

assigned a poly.

$P^x \in \Gamma(x, k', CM)$

to each $x \in S_0$.

WE'VE CARRIED OUT OUR PLAN

FOR S_0.

ONWARD TO THE HIGHER S_ν!

IMPLEMENTING THE PLAN: INDUCTION STEP

Fix ν $(0 \le \nu < \nu_{max})$,
and suppose we have already found
$P^x \in H(x)$ for all $x \in S_\nu$, satisfying

$$\boxed{\begin{array}{l} |\partial^\alpha P^x(x)| \le CM \text{ for } |\alpha| \le m,\ x \in S_\nu \\ \text{and} \\ |\partial^\alpha (P^x - P^y)(x)| \le CM\,\omega(|x-y|) \cdot |x-y|^{m-|\alpha|} \\ \text{for } |\alpha| \le m,\ x, y \in S_\nu \end{array}}$$

(*)

By retaining the P^x for $x \in S_\nu$

and defining new polys

P^x for $x \in S_{\nu+1} \setminus S_\nu$,

we will satisfy $(*)$

with $S_{\nu+1}$ in place of S_ν.

LET'S SEE HOW TO DO THAT.

RECALL THAT $S_{\nu+1}$ ARISES FROM S_ν by SPROUTING x_ν INTO $\hat{S}_\nu$.

BECAUSE $x_\nu \in S_\nu$, WE HAVE ALREADY PRODUCED $\boxed{P^{x_\nu} \in H(x_\nu)}$,

and WE KNOW that

$$\boxed{|\partial^\alpha P^{x_\nu}(x_\nu)| \leq CM \quad \text{for } |\alpha| \leq m.}$$

Moreover, for $y \in \hat{S}_\nu \setminus \{x_\nu\}$,

we have

$$\boxed{|x_\nu - y| \le \operatorname{diam} \hat{S}_\nu \le \delta_0.}$$

Therefore, Tool II applies.

Hence, from $P^{x_\nu} \in H(x_\nu)$, we

obtain a poly. $P^y \in H(y)$ s.t.

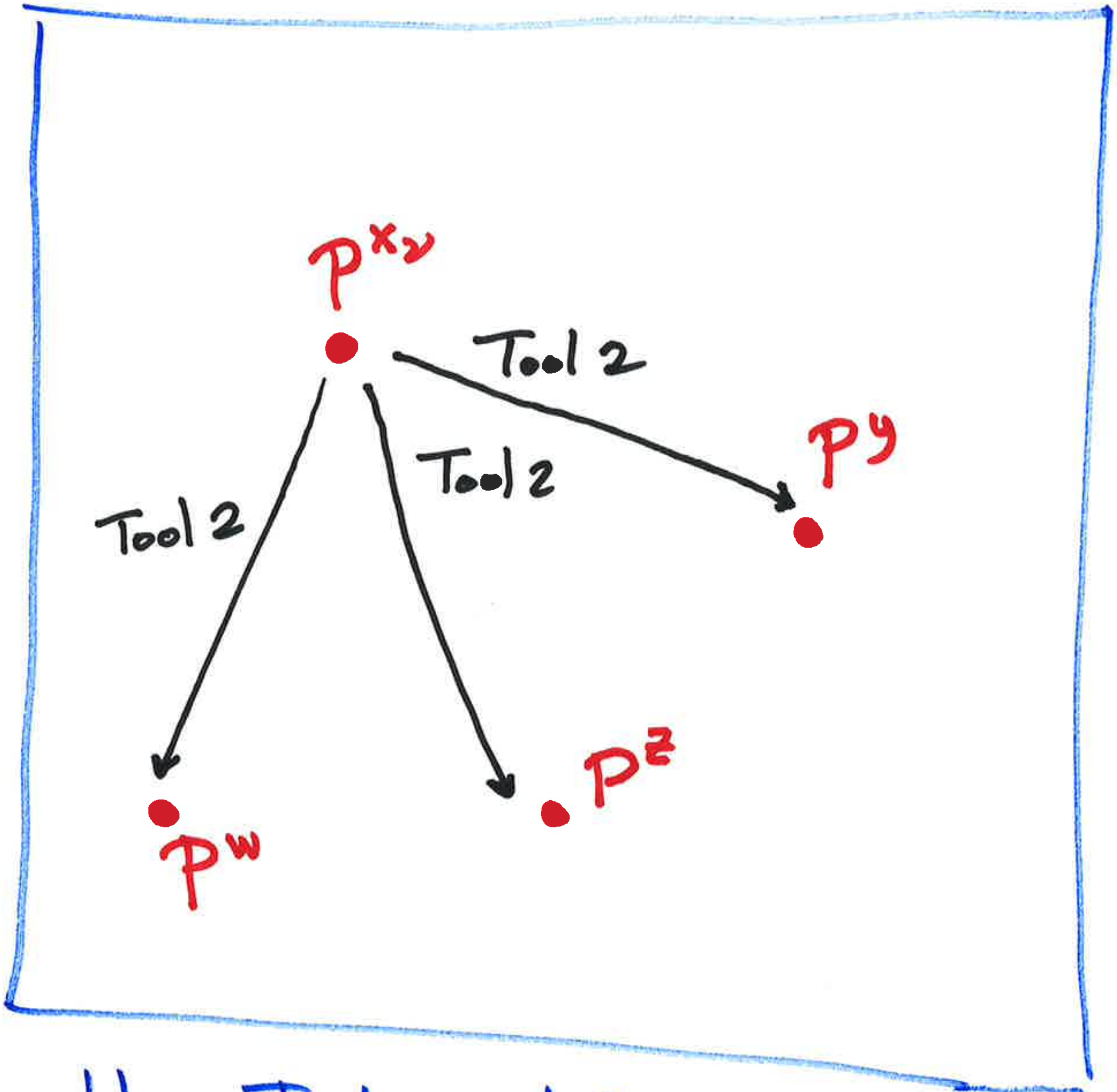
$$|\partial^\alpha (P^{x_\nu} - P^y)(x_\nu)| \le CM\,\omega(|x_\nu - y|) \cdot |x_\nu - y|^{m-|\alpha|}$$

for $|\alpha| \le m$.

We obtain such a $P^y \in H(y)$ for every $y \in \hat{S}_\nu \setminus \{x_\nu\} = S_{\nu+1} \setminus S_\nu$.

$$\boxed{\text{Thus, we have defined } P^x \in H(x) \text{ for every } x \in S_{\nu+1}.}$$

One can check that those P^x satisfy $(*)$, with $S_{\nu+1}$ in place of S_ν.



How Tool 2 defines P^y for $y \in \hat{S}_\nu \setminus \{x_\nu\}$

So our induction on ν

is complete.

Taking $\nu = \nu_{MAX}$, and

recalling that $S_{\nu_{MAX}} = S$,

we obtain the following

result :

We have produced $P^x \in H(x)$ for each $x \in S$, and shown that

$$\boxed{\begin{gathered} |\partial^\alpha P^x(x)| \le CM \\ \text{for } |\alpha| \le m,\ x \in S \end{gathered}}$$

and

$$\boxed{\begin{gathered} |\partial^\alpha (P^x - P^y)(x)| \le \\ CM\,\omega(|x-y|) \cdot |x-y|^{m-|\alpha|} \\ \text{for } |\alpha| \le m,\quad x, y \in S. \end{gathered}}$$

Those are precisely

the estimates arising

in our

CHALLENGE !

ARE WE DONE?

NO!

OUR CHALLENGE

IS TO PRODUCE

POLYS $P^x \in \Gamma(x, k', CM)$

SATISFYING

FAVORABLE ESTIMATES.

For $x \in S_0$, our P^x belong to $\Gamma(x, k', CM)$ as required, but for $x \in S \setminus S_0$, our P^x merely belong to $H(x)$.

Oops!

We are rescued from DISASTER by the following result.

LEMMA ON STABILITY OF Γ's:

For a small enough $\hat{\delta}_0 > 0$, the following holds

Let $x \in E_1$, $P \in \Gamma(x, k', CM)$,

$y \in E_1 \cap B(x, \hat{\delta}_0)$.

If $P' \in H(y)$ and

$|\partial^\alpha (P - P')(x)| \leq CM|x-y|^{m-|\alpha|}$

for $|\alpha| \leq m$,

then $P' \in \Gamma(x, k'', C'M)$.

Here, $k' \gg k'' \gg 1$.

So k gets smaller, but that's OK.

That's proven by applying

HELLY'S THM in the USUAL WAY

+

CONSTANCY of

$\dim H(x)$ and $\dim \pi_x H(x)$

on the lowest stratum E_1.

(LET'S BELIEVE IT — YOU'VE SUFFERED ENOUGH!)

Recall that when we constructed our modulus of continuity, we reserved the right to take δ_0 less than a small positive $\hat{\delta}_0$ to be picked later.

We now pick $\hat{\delta}_0$ as in the Lemma on Stability of Γ's.

Thanks to the

LEMMA on STABILITY OF Γ's,

our P^x $(x \in S \setminus S_0)$

belongs to $\Gamma(x, k'', CM)$

after all.

So we have met the

CHALLENGE

and proven QMT.

WE GOT THROUGH IT!

THANK YOU!